International Journal of Theoretical Physics, Vol. 22, No. 2, 1983

Induced Decay of Positronium and Grasers

Daniel M. Heffernan[1] **and Richard L. Liboff**

Department of Physics and Schools of Electrical Engineering and Applied Physics, Cornell University, Ithaca, New York 14853

Received February 5, 1981

The differential cross section and the total cross section for the stimulated decay of positronium by an incident photon of frequency ω is calculated as a function of the dimensionless variable $\xi = \hbar\omega/mc^2$. For $\xi \gg 1$ the total cross section is found to decrease as ξ^{-2}. We also look at the particular case of positronium in a black-body radiation field. Expressions for the number of induced annihilations per second as functions of the dimensionless ratio mc^2/kT and the number of positronium atoms are obtained. It is found that this rate is proportional to $(kT/mc^2)^2$ for $kT \leqslant mc^2$ and to $(kT/mc^2)\ln(kT/mc^2)$ for $kT \gg mc^2$. The possibility of utilizing induced two-photon decay of positronium as a γ-ray laser at the wavelength $\lambda_C/2$ is examined, where λ_C is the Compton wavelength.

1. INTRODUCTION

1.1. Motivation. Recent interest in the construction of a so-called graser (Ginzberg, 1976) has motivated this brief study of the possibility of inducing a positronium atom (Po) to decay by collision with a photon of some characteristic frequency.

1.2. Review of Two-Photon Decay. Let us first review the known theory (Berestetskii et al., 1971; Pirenne, 1947; Ore et al., 1949; Akhiezer et al., 1965, Stroscio, 1975) for two-photon annihilation of parapositronium ($s = 0$). This process has a decay time $\tau_0 = 1.23 \times 10^{-10}$ sec. The relevant Feynman graphs for this decay are shown in Figure 1. Since the electron and positron in a Po atom have momenta $\sim me^2/\hbar \ll mc$, the standard evaluation for the probability of annihilation assumes the particles to be at rest at the origin.

[1]Permanent address: Dublin Institute for Advanced Studies, Dublin 4, Ireland.

0020-7748/83/0200-0193$03.00/0

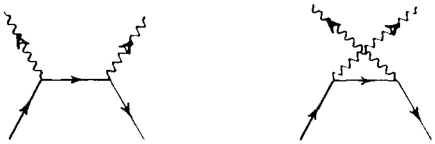

Fig. 1. Feynman graphs for annihilation of parapositronium.

Let $\bar{\sigma}_{2\gamma}$ be the cross section for two-photon annihilation of a free positron–electron pair averaged over their initial spins. The annihilation probability rate is given by

$$\bar{\Gamma} = |\psi(0)|^2 (v\bar{\sigma}_{2\gamma}) \tag{1}$$

where

$$\bar{\sigma}_{2\gamma} = \pi \left(\frac{e^2}{mc^2} \right)^2 \frac{c}{v} \tag{2}$$

is the nonrelativistic cross section for this process and

$$\psi(r) = \frac{1}{(\pi a^3)^{1/2}} e^{-r/a} \tag{3}$$

represents the hydrogenlike ground state wave function of positronium. The length a represents the Bohr radius of Po,

$$a = 2\hbar^2 / me^2 \tag{4}$$

and v is the relative electron–positron speed. Of the four possible total spin states of Po only the $s = 0$ state can annihilate by two-photon decay (Pirenne, 1947). Hence the lifetime of parapositronium is

$$\frac{1}{\tau_0} = 4\bar{\Gamma} = \lim_{v \to 0} 4|\psi(0)|^2 (v\bar{\sigma}_{2\gamma}) \tag{5}$$

which gives the value stated above,

$$\tau_0 = \frac{2\hbar}{mc^2 \alpha^5} = 1.23 \times 10^{-10}\ \text{sec} \tag{6}$$

2. ANALYSIS

2.1. Stimulated Decay of Positronium. We compute the cross section for the scattering of a photon on a positronium atom. The relevant Feynman diagrams for the process $e^+e^-\gamma \to \gamma_1\gamma_2$ are shown in Figure 2. In these diagrams p_-, s_- and p_+, s_+ are the 4-momenta and spins of the electron and positron, respectively, whereas k, ε and k_1, ε_1, k_2, ε_2 are the 4-momenta and polarizations of the incoming and outgoing photons, respectively. [Note that $k=(\omega, \mathbf{k})$ and $p=(E, \mathbf{p})$ and we have set $\hbar = c = 1$.]

The matrix element for this process is

$$\begin{aligned} M_{fi} = -ie^3\bar{v}(p_+, s_+)\Bigg(& \not{\varepsilon}_1 \frac{1}{\not{p}_- + \not{k} - \not{k}_2 - m}\not{\varepsilon}\frac{1}{\not{p}_- - \not{k}_2 - m}\not{\varepsilon}_2 \\ & + \not{\varepsilon}_2 \frac{1}{\not{p}_- - \not{k} - \not{k}_1 - m}\not{\varepsilon}\frac{1}{\not{p}_- - \not{k}_1 - m}\not{\varepsilon}_1 \\ & + \not{\varepsilon}_1 \frac{1}{\not{p}_- + \not{k} - \not{k}_2 - m}\not{\varepsilon}_2\frac{1}{\not{p}_- + \not{k} - m}\not{\varepsilon} \\ & + \not{\varepsilon}_2 \frac{1}{\not{p}_- + \not{k} - \not{k}_1 - m}\not{\varepsilon}_1\frac{1}{\not{p}_- + \not{k} - m}\not{\varepsilon} \\ & + \not{\varepsilon} \frac{1}{\not{p}_- - \not{k}_1 - \not{k}_2 - m}\not{\varepsilon}_1\frac{1}{\not{p}_- - \not{k}_2 - m}\not{\varepsilon}_2 \\ & + \not{\varepsilon} \frac{1}{\not{p}_- - \not{k}_1 - \not{k}_2 - m}\not{\varepsilon}_2\frac{1}{\not{p}_- - \not{k}_1 - m}\not{\varepsilon}_1\Bigg) u(p_-, s_-) \end{aligned} \tag{7}$$

In this expression v and u are positron and electron spinors, respectively, which satisfy the Dirac equations (Bjorken et al., 1964)

$$(\not{p}_+ + m)v(p_+, s_+) = 0 \tag{8}$$

$$(\not{p}_- - m)u(p_-, s_-) = 0 \tag{9}$$

Slashed 4-vectors follow the standard convention of representing the 4-inner product with the γ matrices (Bjorken et al., 1964). We work in the transverse gauge such that the 4-polarization vector ε is given by

$$\varepsilon = (0, \boldsymbol{\varepsilon}) \tag{10}$$

and

$$\boldsymbol{\varepsilon}\cdot \mathbf{k} = 0 \tag{11}$$

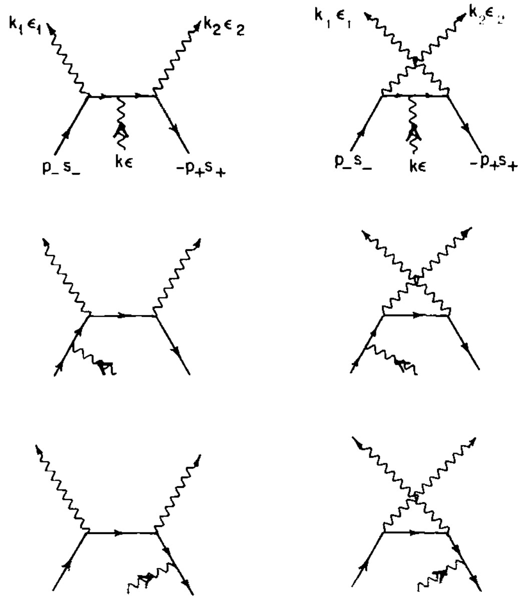

Fig. 2. Feynman graphs for induced decay of positronium.

Similarly for the outgoing photons. Throughout we use the conventions of Berestetskii et al. (1971). That is, fermion wavefunctions are normalized as

$$\psi_e = \frac{1}{(2E_-)^{1/2}} u(p_-, s_-) e^{-ip_- \cdot x} \tag{12}$$

for the electron, and as

$$\psi_p = \frac{1}{(2E_+)^{1/2}} v(p_+, s_+) e^{ip_+ \cdot x} \tag{13}$$

for the positron. The Fourier decomposition of the photon field is given by

$$A_k^\mu = (4\pi)^{1/2} \frac{\varepsilon^\mu}{(2\omega)^{1/2}} e^{-ik \cdot x} \tag{14}$$

where $x = (t, \mathbf{r})$.

Again we assume that the speed of the electron relative to the positron in positronium is small compared to c and set $p_- = p_+ = 0$. The squared modulus of the amplitude, averaged over the polarizations of the electron and positron and incident photon, and summed over the polarizations of the outgoing photons, then has the value

$$\overline{|M_{fi}|^2} = 4(4\pi)^3 e^6 \left[\left(\frac{m-\omega_1}{\omega\omega_2} \right)^2 + \left(\frac{m-\omega_2}{\omega\omega_1} \right)^2 + \left(\frac{m+\omega}{\omega_1\omega_2} \right)^2 \right] \quad (15)$$

In deriving this result we have used the fact that this process can be obtained by cross-channeling the three-photon decay of orthopositronium (Ore et al., 1949).

This value of $\overline{|M_{fi}|^2}$ completes the following expression for the differential cross section (Berestetskii et al., 1971):

$$d\sigma = (2\pi)^4 \delta^4(k_1 + k_2 - k - p) \times \frac{\overline{|M_{fi}|^2}}{2E_+ 2E_- 2\omega} |\psi(0)|^2 \frac{d^3k_1}{(2\pi)^3 \cdot 2\omega_1} \frac{d^3k_2}{(2\pi)^3 \cdot 2\omega_2} \quad (16)$$

where

$$p = p_- + (-p_+) \quad (17)$$

[Here we have taken the positronium atom to be at rest and hence the incident flux of photons per unit volume is c, which in our units is just 1. (See Akhiezer and Berestetskii, 1965).] In this expression the positronium wave functions have been projected onto the free electron and positron wave functions to account for the fact that positronium is a bound system of an electron and a positron. We assume that the Po atom is in its ground state so that with equation (3), $\psi(0) = 1/(\pi a^3)^{1/2}$. Again assuming that the electron and positron are at rest and integrating over k_2 we obtain

$$\frac{d\sigma}{d\Omega_1} = \frac{1}{16(2\pi)^2} \frac{|\psi(0)|^2}{m^2\omega} \int_0^{2m+\omega} \omega_1 \, d\omega_1 \overline{|M_{fi}|^2} \times \delta\{4m[m+\omega] - 4\omega_1[m + \omega \sin^2(\theta/2)]\} \quad (18)$$

where $d^2k_1 = k_1^2 dk_1 d\Omega_1$ and θ is the angle between $\mathbf{k}_1$ and $\mathbf{k}$. Completing the integration gives

$$\frac{d\sigma}{d\Omega_1} = \pi e^6 \frac{|\psi(0)|^2\omega_1^2}{m^3(m+\omega)\omega} \times \left[\frac{(m-\omega_1)^2}{\omega^2(2m+\omega-\omega_1)^2} + \frac{(m+\omega-\omega_1)^2}{\omega^2\omega_1^2} + \frac{(m+\omega)^2}{\omega_1^2(2m+\omega-\omega_1)^2}\right] \tag{19}$$

where

$$\omega_1 = \frac{m(1+\omega/m)}{1+(\omega/m)\sin^2(\theta/2)} \tag{20}$$

Converting to wavelength, $\lambda = 2\pi c/\omega$, we note that the latter equation may be cast in a form closely resembling the Compton-effect relation. There results,

$$\lambda - \lambda_1 = \lambda_C\left(\frac{\lambda_1\lambda}{\lambda_C^2} - \sin^2\frac{\theta}{2}\right) \tag{20$'$}$$

where λ_C has been written for the Compton wavelength. If incident and emitted photons are all of equal wavelength, then momentum conservation indicates that $\theta = \pi/3$. In this case (20′) gives

$$\lambda = \lambda_C/2$$

which is the Compton wavelength relevant to positronium. This result agrees with energy conservation under the same constraint. At this wavelength, alternate induced decays in a plane would result in superposed colinear radiation with the incident flux.

It is noted in passing that (20′) may be obtained directly from kinematics. Returning to the mainstream of the analysis, we rewrite $d\sigma/d\Omega$ in the form

$$\frac{d\sigma}{d\Omega_1} = \frac{1}{8}\frac{e^{12}\omega_1^2}{m^2(1+\omega/m)\omega/m}\left[\frac{(m-\omega_1)^2}{\omega^2(2m+\omega-\omega_1)^2} + \frac{(m+\omega-\omega_1)^2}{\omega^2\omega_1^2} + \frac{(m+\omega)^2}{\omega_1^2(2m+\omega-\omega_1)^2}\right] \tag{21}$$

In full-dimensional units this expression becomes

$$\frac{d\sigma}{d\Omega_1}=\frac{1}{8}\alpha^6\frac{\hbar^2}{m^2c^2}\frac{\left(\hbar\omega_1/mc^2\right)^2}{\left(\hbar\omega/mc^2\right)\left(1+\hbar\omega/mc^2\right)}$$

$$\times\left[\frac{\left(1-\hbar\omega_1/mc^2\right)^2}{\left(\hbar\omega/mc^2\right)^2\left(2+\hbar\omega/mc^2-\hbar\omega_1/mc^2\right)^2}\right.$$

$$+\frac{\left(1+\hbar\omega/mc^2-\hbar\omega_1/mc^2\right)^2}{\left(\hbar\omega/mc^2\right)^2\left(\hbar\omega_1/mc^2\right)^2}$$

$$\left.+\frac{\left(1+\hbar\omega/mc^2\right)^2}{\left(\hbar\omega_1/mc^2\right)^2\left(2+\hbar\omega/mc^2-\hbar\omega_1/mc^2\right)^2}\right] \qquad (22)$$

where $\alpha=e^2/\hbar c\approx 1/137$ is the fine-structure constant.

The evaluation of σ is simplified by setting

$$\begin{aligned}&(a)\quad x\equiv\sin^2\frac{\theta}{2}\\&(b)\quad \xi\equiv\frac{\omega}{m}\\&(c)\quad \eta\equiv\xi(\xi+2)\end{aligned} \qquad (23)$$

(where we have reverted to units in which $\hbar=c=1$).

With these substitutions we obtain

$$\sigma=4\pi\int_0^1 dx\, g(x) \qquad (24)$$

where

$$g(x)=\frac{1}{8}\frac{e^{12}}{m^2}\frac{(1+\xi)}{\xi}\left[\frac{(x-1)^2}{(1+\xi x)^2(1+\eta x)^2}+\frac{x^2}{(1+\xi x)^2}+\frac{(1+\xi x)^2}{(1+\eta x)^2}\right] \qquad (25)$$

The three integrals which enter the evaluation of σ are

$$I_1=\int_0^1 \frac{dx(x-1)^2}{(1+\xi x)^2(1+\eta x)^2}=\frac{1}{\xi^3}\left[\frac{\xi(\xi+2)(\xi+1)}{(\xi+1)^2}-2\ln(\xi+1)\right] \quad (26)$$

$$I_2=\int_0^1 dx\frac{x^2}{(1+\xi x)^2}=\frac{1}{\xi^3}\left[\frac{\xi(\xi+2)}{\xi+1}-2\ln(\xi+1)\right] \quad (27)$$

$$I_3=\int_0^1 dx\frac{(1+\xi x)^2}{(1+\eta x)^2}=\frac{2}{\xi(\xi+2)^3}[\xi(\xi+2)+2(\xi+1)\ln(\xi+1)] \quad (28)$$

Substituting these terms in equation (24), we obtain

$$\sigma(\xi)=\frac{\pi e^{12}}{m^2}\left\{\frac{1}{\xi^4(\xi+2)^3}\left[\xi(\xi+2)(2\xi^3+7\xi^2+12\xi+8)\right.\right.$$
$$\left.\left.-2(\xi+1)(5\xi^2+12\xi+8)\ln(\xi+1)\right]\right\} \quad (29)$$

or, equivalently,

$$\sigma(\xi)=\frac{\pi e^{12}}{m^2}f(\xi) \quad (30)$$

which serves to define the function $f(\xi)$. Note that $\sigma(\xi)\sim 1/\xi$ as $\xi\to 0$ which is the expected infrared singularity (Weisskopf, 1949; Feynman, 1948; Schwinger, 1948).

A graph of $f(\xi)$ is shown in Figure 3. It decreases monotonically and tends to zero like ξ^{-2} as $\xi\to\infty$.

In dimensional units σ appears as

$$\sigma(\xi)=\pi\alpha^6\frac{\hbar^2}{m^2c^2}f(\xi) \quad (31)$$

where $\xi\equiv\hbar\omega/mc^2$ and α is the fine-structure constant, or, equivalently,

$$\sigma(\xi)=\pi\alpha^4 r_0^2 f(\xi) \quad (32)$$

where $r_0=e^2/mc^2$, the classical electron radius.

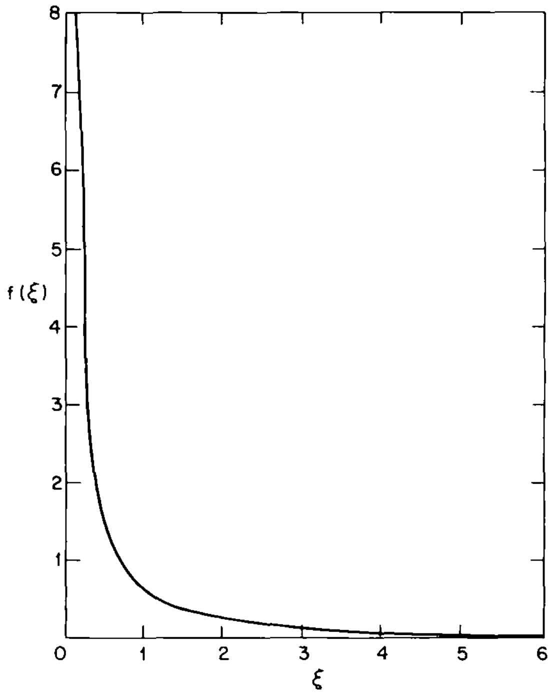

Fig. 3. The function of $f(\xi)$.

2.2. Positronium in a Black-Body Radiation Field. Suppose we have an aggregate of Po atoms immersed in a black-body radiation field. The number of annihilations per second is given by

$$\frac{dN}{dt} = -N\int_0^{\infty} d\omega \frac{dn(\omega)}{d\omega}\langle c\sigma(\omega)\rangle \tag{33}$$

where N is the number of positronium atoms, $n(\omega)$ is the number of photons of frequency ω, and $\sigma(\omega)$ is the cross section for photon scattering off positronium. The average is over the possible velocities of the photon with respect to the positronium atom (Smirnov, 1977), which is trivial since we assume the positronium atom to be at rest and c is the velocity of the photons which is constant. Annihilation of positronium atoms due to spontaneous decay may be incorporated into the latter equation by adding an extra term $-N/\tau_0$, where $\tau_0 \approx 10^{-10}$ sec. represents the lifetime of a positronium atom (see Section 1.2). It follows that effects of induced decay on the density of positronium atoms in a radiation field will be influenced by the field provided:

$$\int_0^{\infty} d\omega \frac{dn(\omega)}{d\omega} \langle c\sigma(\omega) \rangle > \frac{1}{\tau_0} \approx 10^{10}\ \text{Hz} \tag{34}$$

The number of photons per unit volume in the frequency range $\omega, \omega + d\omega$ is given by

$$\frac{dn(\omega)}{d\omega} = \frac{1}{\pi^2 c^3} \frac{\omega^2}{\exp(\hbar\omega/k_B T) - 1} \tag{35}$$

Hence, with (33) we obtain

$$\frac{dN}{dt} = -\frac{1}{\pi^2 c^2} N \int_0^{\infty} d\omega \frac{\omega^2 \sigma(\omega)}{\exp(\hbar\omega/k_B T) - 1}$$

i.e.,

$$\frac{dN}{dt} = -N\left(\frac{\hbar}{mc^2}\right)^2 \frac{\alpha^6}{\pi} \int_0^{\infty} d\omega \frac{\omega^2 f(\hbar\omega/mc^2)}{\exp(\hbar\omega/k_B T) - 1} \tag{36}$$

Again writing this expression in terms of the dimensionless parameter $\xi = \hbar\omega/mc^2$, the latter equation becomes

$$\frac{dN}{dt} = -\frac{mc^2}{\hbar} \frac{\alpha^6}{\pi} N \int_0^{\infty} d\xi \frac{\xi^2 f(\xi)}{e^{a\xi} - 1} \tag{37}$$

where,

$$a = \frac{mc^2}{k_B T} \tag{38}$$

The number of annihilations per second at the dimensionless frequency ξ is

$$\frac{dN}{dt\, d\xi} = -\frac{1}{\pi}\left(\frac{mc^2}{\hbar}\right) \alpha^6 N \frac{\xi^2 f(\xi)}{e^{a\xi} - 1} \tag{39}$$

Let us return to equation (37). As $f(\xi)$ is rather complicated [see equation (29)], the integration over ξ is somewhat difficult. Let us approximate $f(\xi)$ by

$$f_1(\xi) \approx \frac{1}{\xi + \xi^2} \tag{40}$$

Note that,

$$f_1(\xi) \sim \frac{1}{\xi} \quad \text{for } \xi \to 0$$

and

$$f_1(\xi) \sim \frac{1}{\xi^2} \quad \text{for } \xi \to \infty$$

so $f_1(\xi)$ has the right asymptotic limits. It will serve to indicate how the rate of annihilation depends on a. We have now to calculate the integral

$$I(a) \equiv \int_0^\infty d\xi \frac{\xi^2 f_1(\xi)}{e^{a\xi} - 1} \tag{41}$$

Substituting (40) in (41),

$$I(a) = \int_0^\infty d\xi \frac{\xi}{(\xi+1)} \frac{1}{(e^{a\xi} - 1)} \tag{42}$$

we obtain

$$I(a) = \sum_{n=1}^{\infty} \left[\frac{1}{na} + e^{na} E_1(-na) \right] \tag{43}$$

where (Abramowitz et al., 1972; Gradshteyn et al., 1965)

$$E_1(-\mu) = -\int_1^\infty \frac{e^{-\mu y}}{y} dy \tag{44}$$

Simple expressions for $I(a)$ may be obtained in the physically interesting limits $a \gg 1$ and $a \ll 1$. Let $u = a\xi$; then

$$I(a) = \frac{1}{a} \int_0^\infty \frac{u\,du}{(u+a)(e^u - 1)} \tag{45}$$

For a given value of a,

$$I(a) = \frac{1}{a} \int_{\substack{0 \\ u \leqslant a}}^{a} \frac{u\,du}{a(1+u/a)(e^u - 1)} + \frac{1}{a} \int_{\substack{a \\ u \geqslant a}}^{\infty} \frac{u\,du}{u(1+a/u)(e^u - 1)} \tag{46}$$

i.e.,

$$I(a)=\frac{1}{a^2}\int_{\substack{0\\u\leqslant a}}^{a} u\left(1-\frac{u}{a}+\cdots\right)\frac{du}{e^u-1}+\frac{1}{a}\int_{\substack{a\\u\geqslant a}}^{\infty}\left(1-\frac{a}{u}+\cdots\right)\frac{du}{e^u-1} \tag{47}$$

In the limit $a \gg 1$, i.e., $k_B T \ll mc^2$, the second term is negligible and the first term becomes

$$I(a)\approx\frac{1}{a^2}\int_0^{\infty}\frac{u\,du}{e^u-1}+0\left(\frac{1}{a^3}\right)$$

i.e.,

$$I(a)\approx\frac{\pi^2}{6a^2} \tag{48}$$

Hence

$$\frac{dN}{dt}=-\frac{\pi}{6}\frac{\alpha^6}{a^2}\frac{mc^2}{\hbar}N \tag{49}$$

For positronium (48) becomes

$$\frac{dN}{dt}=-\frac{\pi}{6}\frac{\bar{\omega}}{a^2}N, \qquad \bar{\omega}\equiv\alpha^6\frac{mc^2}{\hbar} \tag{50}$$

Therefore in the limit $a \gg 1$ ($k_B T \ll mc^2$), the number of induced annihilations per second is proportional to $(k_B T/mc^2)^2$ and N. Note, however, this limit is physically uninteresting since in this range positronium decays spontaneously. By condition (34) we see that $10^9/a^2 > 10^{10}$ for induced decay to dominate, that is, $a<1$ ($k_B T > mc^2$).

In the physically interesting limit $a \ll 1$ (i.e., $k_B T \gg mc^2$), the dominant term is

$$I(a)\approx\frac{1}{a}\int_a^{\infty}\frac{du}{e^u-1}=-\frac{\ln(1-e^{-a})}{a} \tag{51}$$

that is,

$$I(a)\approx-\frac{\ln a}{a}=\frac{|\ln a|}{a} \tag{52}$$

Hence for $k_B T \gg mc^2$,

$$\frac{dN}{dt} = -\frac{\alpha^6}{\pi}\frac{mc^2}{\hbar}N\frac{|\ln a|}{a} \tag{53}$$

For positronium,

$$\frac{dN}{dt} = -\frac{\bar{\omega}}{\pi}\frac{|\ln a|}{a}N \tag{54}$$

In this limit the number of induced annihilations per second is proportional to $(k_B T/mc^2)\ln(k_B T/mc^2)$ and to N. This is important since in this limit induced decay dominates spontaneous decay.

Actually, (49) is a good approximation even when $a \approx 1$. To see this note that we can approximate $f(\xi)$ when $a \approx 1$ by

$$f_2(\xi) = \frac{(1/a)(1-e^{-a\xi})}{\xi^2} \tag{55}$$

$f_2(\xi)$ has the correct asymptotic limits

$$f_2(\xi) \approx \begin{cases} \dfrac{1}{\xi} & \text{for } \xi \ll 1 \\ \dfrac{1}{\xi^2} & \text{for } \xi \gg 1 \end{cases}$$

and $a \approx 1$. Therefore,

$$I(a) = \int_0^\infty d\xi \frac{\xi^2 f_2(\xi)}{e^{a\xi}-1} = \frac{1}{a^2} \tag{56}$$

It follows that when $a \approx 1$ the number of induced annihilations per second is proportional to $(k_B T/mc^2)^2$ and to N.

In summary we note that for $a \gtrsim 1$ the number of induced annihilations per second is proportional to $1/a^2$ and to N, whereas when $a \ll 1$ the number is proportional to $|\ln a|/a$.

3. CONCLUSIONS

We have calculated the differential cross section and the total cross section for the scattering of a photon off a positronium atom. The cross section was found to be a monotonically decreasing function of the frequency of the incident photon. Expressions were obtained for the number of induced annihilations per second of positronium atoms in a black-body radiation field as a function of the black-body temperature. It was also shown that at wavelength $\lambda_C/2$, the possibility exists of utilizing induced two-photon decay of positronium as a γ-ray laser.

ACKNOWLEDGMENTS

Helpful discussions with our colleagues G. Bhanot, K. C. Liu, M. Burns, A. Garg, B. Lindquist, S. Rudaz, and D. Hammer are gratefully acknowledged. This research was supported in part by Contract No. AFOSR 78-3574.

REFERENCES

Abramowitz, M. and Stegun, I. (1972). *Handbook of Mathematical Functions*. Dover, New York.
Akhiezer, A., and Berestetskii, V. (1965). *Quantum Electrodynamics*. Interscience Publishers, New York.
Berestetskii, V., Lifshitz, E., and Pitaeskii, L. (1971). *Relativistic Quantum Theory*, Part 1. Pergamon Press, Oxford.
Bjorken, J., and Drell, S. (1964). *Relativistic Quantum Mechanics*. McGraw-Hill, New York.
Feynman, R. P. (1948). *Physical Review*, **74**, 939, 1430.
Ginzburg, V. (1976). *Key Problems of Physics and Astrophysics*. MIR Publishers, Moscow.
Gradshteyn, I., and Ryshik, I. (1965). *Table of Integrals Series and Products*, Academic Press, New York.
Ore, A., and Powell, J. (1949). *Physical Review*, **75**, 1696.
Pirenne, J. (1947). *Archives des Sciences Physiques et Naturelles*, **29**, 207.
Schwinger, J. (1948). *Physical Review*, **73**, 415; **74**, 1439; **75**, 651.
Smirnov, B. (1977). *Introduction to Plasma Physics*, MIR Publishers, Moscow.
Stroscio, M. A. (1975). *Physics Reports, Section C*, **22**, 215.
Weisskopf, V. (1949). *Reviews of Modern Physics*, **21**, 2, 305.